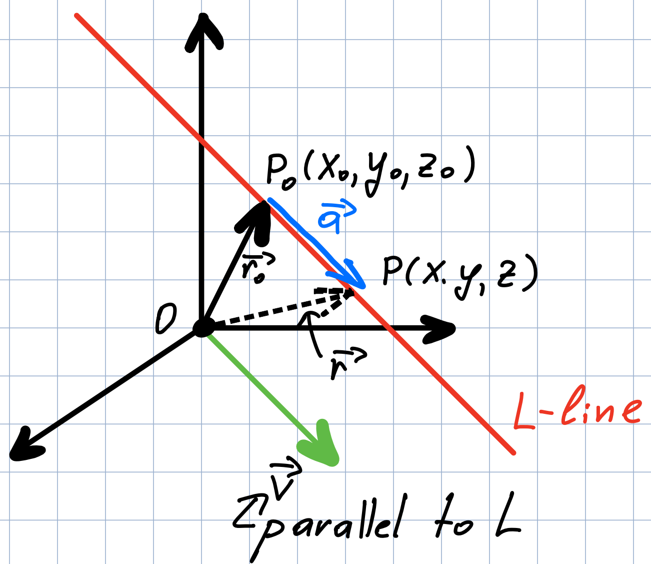
Last time: Lines

$$\boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$$

– vector equation of L

$t \in \mathbb{R}$ parameter



• Let $\vec{v} = \langle a, b, c \rangle$

then in components:

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

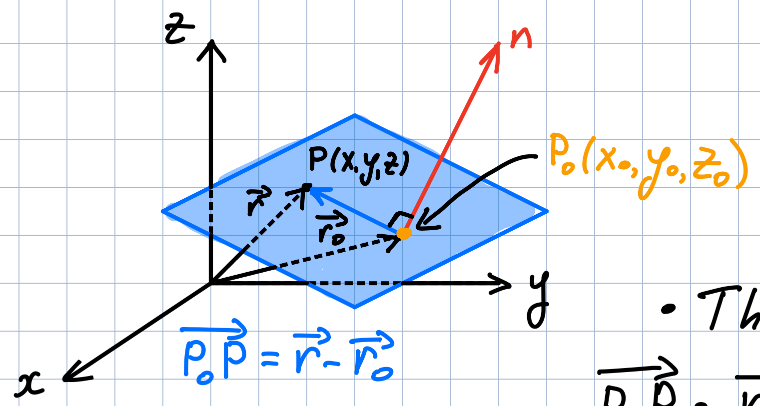
$$\boxed{\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}}$$

– parametric equations of the line L through point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle a, b, c \rangle$

$$(t =) \quad \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

– symmetric equations of L

Today: Planes



- A plane is defined by a point P_0 and an orthogonal vector $\vec{n}$ ("the normal vector")

- Then for any point P on the plane,

$\overrightarrow{P_0P} \cdot \vec{n} = 0 \quad \Rightarrow \quad \boxed{\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0}$

or $\boxed{\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}}$

vector eq. of the plane

Reminder:

$\vec{a} = \langle a_1, a_2, a_3 \rangle \qquad \vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

vectors are orthogonal $\Leftrightarrow$ $\vec{a} \cdot \vec{b} = 0$

If $\vec{n} = \langle a, b, c \rangle$, $\vec{r} = \langle x, y, z \rangle$, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$, then eq. of the plane is

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

– scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$

Ex: a) Find an equation of the plane through the point $(1,-2,4)$ with normal vector $n = \langle 2,3,4 \rangle$.

b) Find the intercepts and sketch the plane.

Solution:

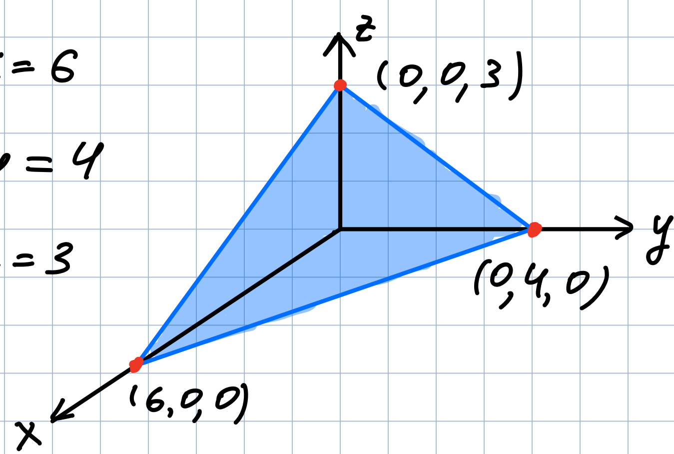
$$2(x-1) + 3(y+2) + 4(z-4) = 0$$

or $\quad 2x + 3y + 4z = 2 - 6 + 16 = 12$

x-intercept: $\quad y = z = 0 \quad \Rightarrow \quad 2x = 12 \quad \Rightarrow \quad x = 6$

y-intercept: $\quad x = z = 0 \quad \Rightarrow \quad 3y = 12 \quad \Rightarrow \quad y = 4$

z-intercept: $\quad x = y = 0 \quad \Rightarrow \quad 4z = 12 \quad \Rightarrow \quad z = 3$



Rmk: $\boxed{ax + by + cz + d = 0}$ - linear equation of the plane

$$d = -(ax_0 + by_0 + cz_0)$$

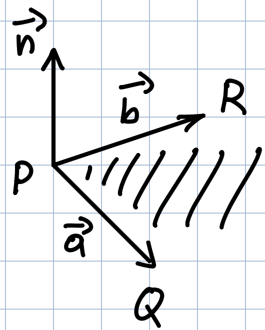
$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$ - scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$

Ex: $P(-1,2,5)$, $Q(0,4,8)$, $R(1,2,0)$

Find an equation of the plane through P, Q, R

Solution: $\vec{a} = \overrightarrow{PQ} = \langle 1, 2, 3 \rangle \qquad \vec{b} = \overrightarrow{PR} = \langle 2, 0, -5 \rangle$

normal vector: $\vec{n} = \vec{a} \times \vec{b}$



$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 0 & -5 \end{vmatrix} = \langle 2(-5) - 0(3),\ 3(2) - 1(-5),\ 1(0) - 2(2) \rangle$$

$$= \langle -10, 11, -4 \rangle$$

Take $P_0 = P$, then eq. of the plane:

$$-10(x+1) + 11(y-2) - 4(z-5) = 0$$

or $\quad -10x + 11y - 4z - 12 = 0$

$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$

Ex: Find the intersection point of plane with the line

$x = -t,\ y = t - 1,\ z = 6t - 2$

Sol: $-10(-t) + 11(t-1) - 4(6t-2) - 12 = 0$

$$-3t - 11 + 8 - 12 = 0$$

$$3t = -15 \Rightarrow t = -5$$

$(x, y, z) = (5, -6, -32)$

Check:

$$\underbrace{-10(5) + 11(-6)}_{-116} \underbrace{-4(-32)}_{+128} - 12 = 0$$

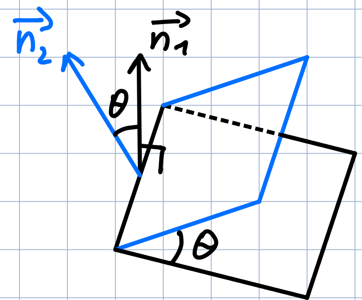
$$-12 = 0$$

- Two planes are parallel if their normal vectors are parallel.

<u>Ex:</u> $-10x + 11y - 4z - 12 = 0$ and $20x - 22y + 8z + 1 = 0$ are parallel.

$\vec{n}_1 = \langle -10, 11, -4 \rangle$ $\qquad$ $\vec{n}_2 = \langle 20, -22, 8 \rangle$

- Two non-parallel planes intersect in a line



angle between planes
= angle between normal vectors

<u>Ex:</u> two planes $x + y + 2z = 0$, $-y + z = 1$

a) Find the angle between them b) Find a sym. eq. of the intersect. line L

<u>Sol:</u> (a) $\vec{n}_1 = \langle 1, 1, 2 \rangle$

$\vec{n}_2 = \langle 0, -1, 1 \rangle$

$$\cos\theta = \frac{1(0) + 1(-1) + 2(1)}{\sqrt{1+1+4}\,\sqrt{0+1+1}} = \frac{1}{\sqrt{12}} \Rightarrow \theta = \arccos\left(\frac{1}{\sqrt{12}}\right)$$

<u>Corollary:</u>
For $\vec{a}, \vec{b}$ nonzero vectors
$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

(b) Need a point on L: set $z=0$

$$\begin{cases} x+y+2(0)=0 \\ -y+0=1 \end{cases} \Rightarrow \begin{matrix} y=-1 \\ x=1 \end{matrix} \Rightarrow$$

$P_0(1,-1,0)$

Direction vector $\vec{v}$ is orthogonal to both $\vec{n_1}$ and $\vec{n_2}$, take

$$\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix} = (1(1)-2(-1))\vec{i} + (2(0)-1(1))\vec{j} + (1(-1)-1(0))\vec{k}$$
$$= 3\vec{i} - \vec{j} - \vec{k} = \langle 3,-1,-1 \rangle$$

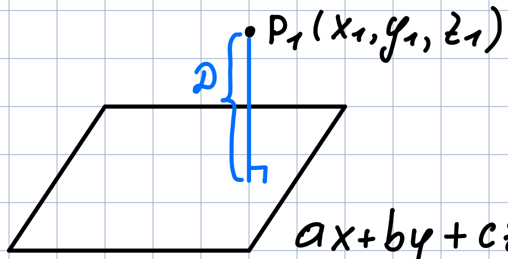
Sym. eq. for L: $\frac{x-1}{3} = \frac{y+1}{-1} = \frac{z}{-1}$

Rmk: We can obtain same line as intersection of planes

$$\frac{x-1}{3} = \frac{y+1}{-1} \quad \text{and} \quad \frac{y+1}{-1} = \frac{z}{-1}$$

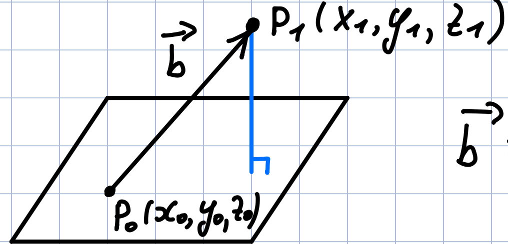
Distances

Distance from a point to a plane.



$ax + by + cz + d = 0$

Let P_0 be any point in the plane



$\vec{b} = \overrightarrow{P_0P_1}$

$\vec{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

$$\boxed{\text{Distance}} \quad D = |\text{comp}_{\vec{n}} \overrightarrow{P_0P_1}| = \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$\boxed{= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Ex: Find the distance from the point $(1,1,1)$ to the plane $x + y + 2z = 0$

Sol:

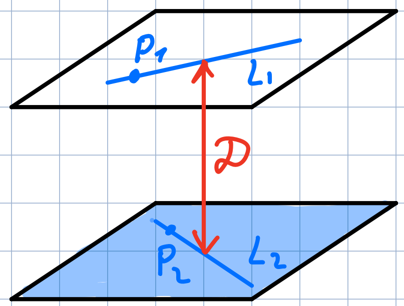
$$D = \frac{|1 + 1 + 2|}{\sqrt{1 + 1 + 2^2}} = \frac{4}{\sqrt{6}}$$

Ex: Skew lines $L_1: x = 1+t, \; y = -2+3t, \; z = 4-t$

$L_2: x = 2s, \; y = 3+s, \; z = -3+4s$

Find distance between L_1, L_2

Solution:



Common normal vector to both planes:

$$\vec{n} = \vec{v_1} \times \vec{v_2} = \langle 1, 3, -1 \rangle \times \langle 2, 1, 4 \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = (3(4) - 1(-1))\vec{i} + ((-1)2 - (1)4)\vec{j}$$

$$+ (1(1) - 2(3))\vec{k} = \langle 13, -6, -5 \rangle$$

P_2: $s = 0 \Rightarrow P_2(2(0), 3+0, -3+4(0)) = (0, 3, -3)$

plane: $13x - 6(y-3) - 5(z+3) = 0$

or $13x - 6y - 5z + 3 = 0$

P_1: $t = 0 \Rightarrow P_1(1, -2, 4)$

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0{,}53$$